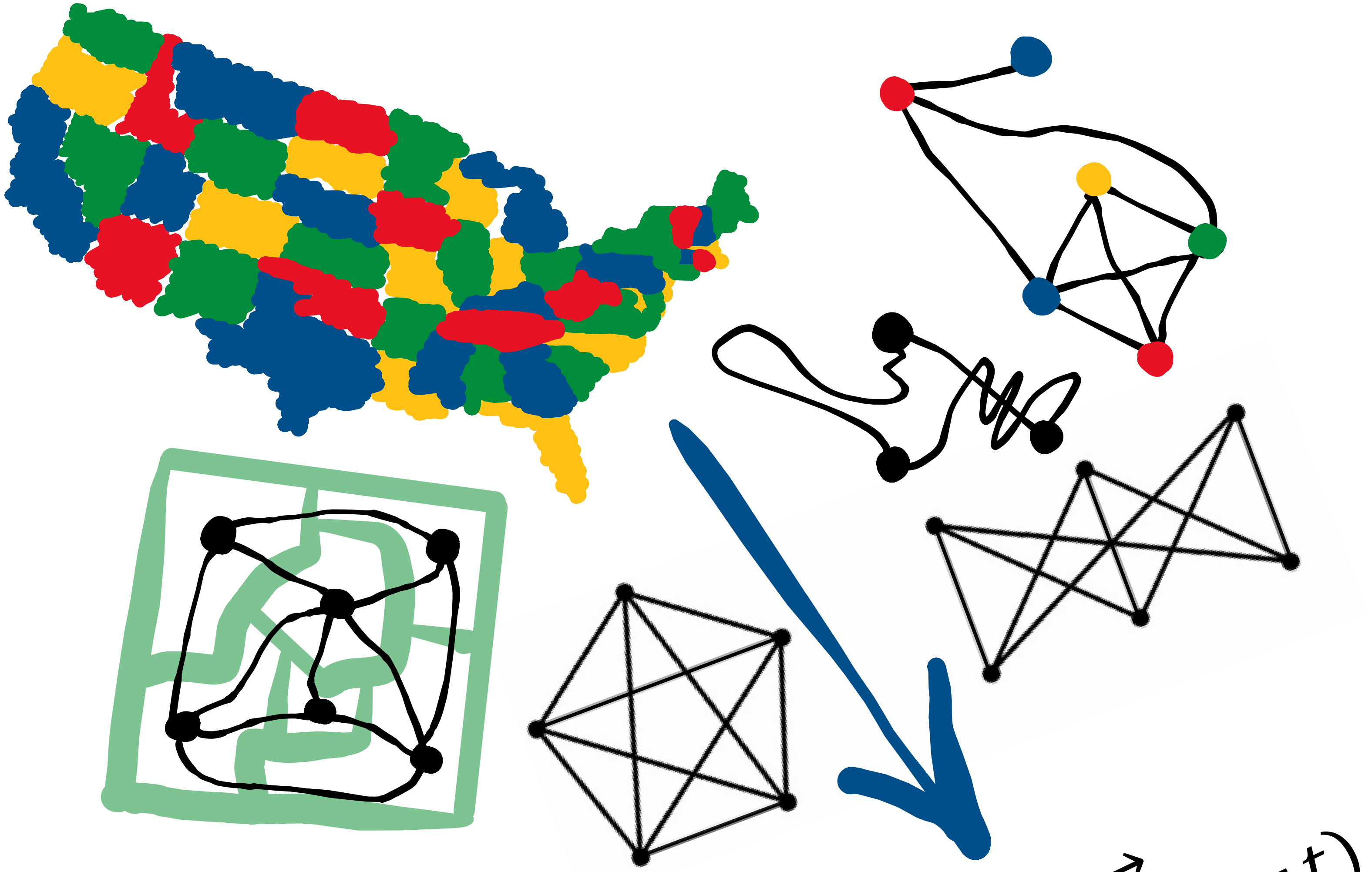
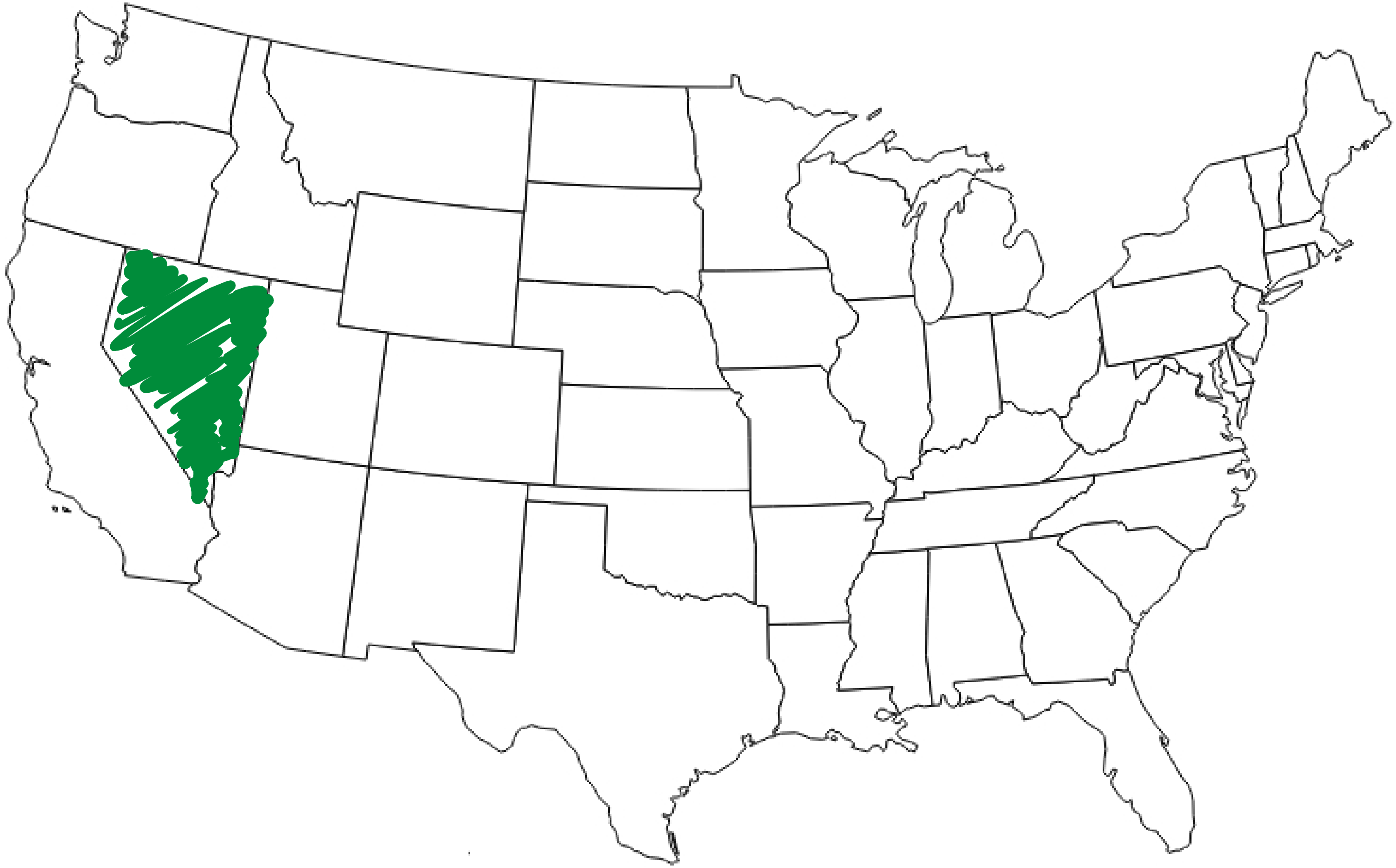
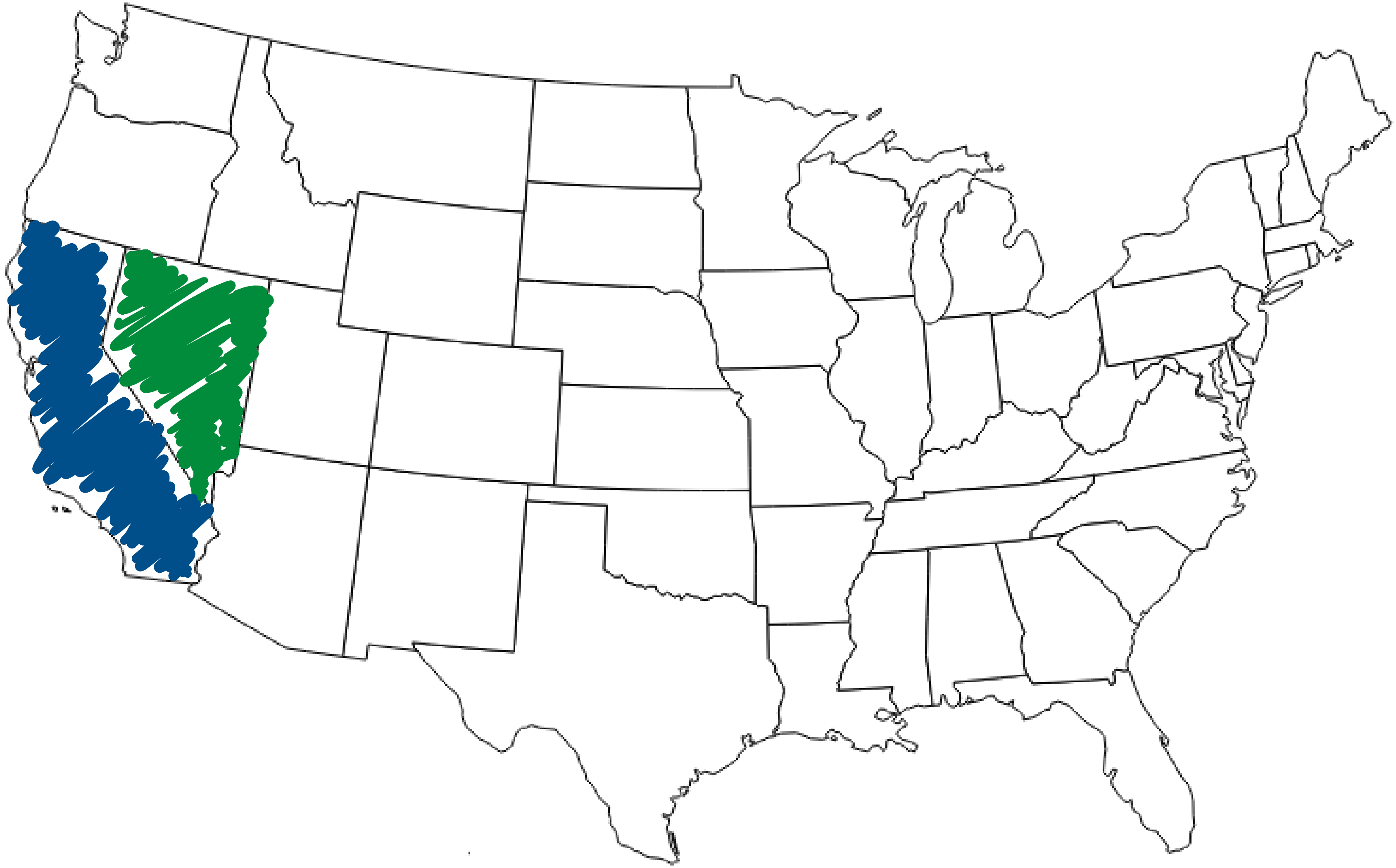
FROM COLORING MAPS TO HADWIGER'S CONJECTURE

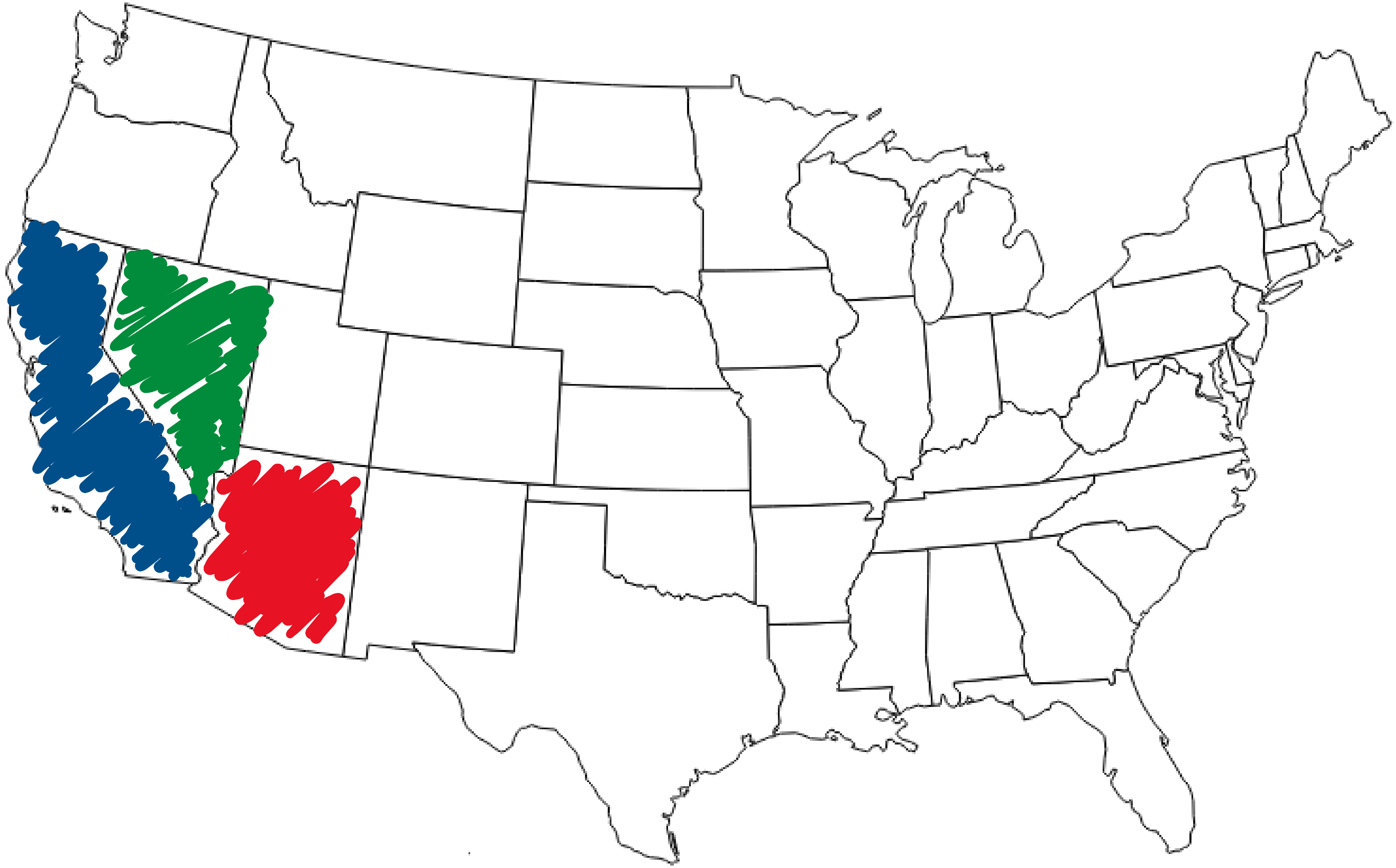


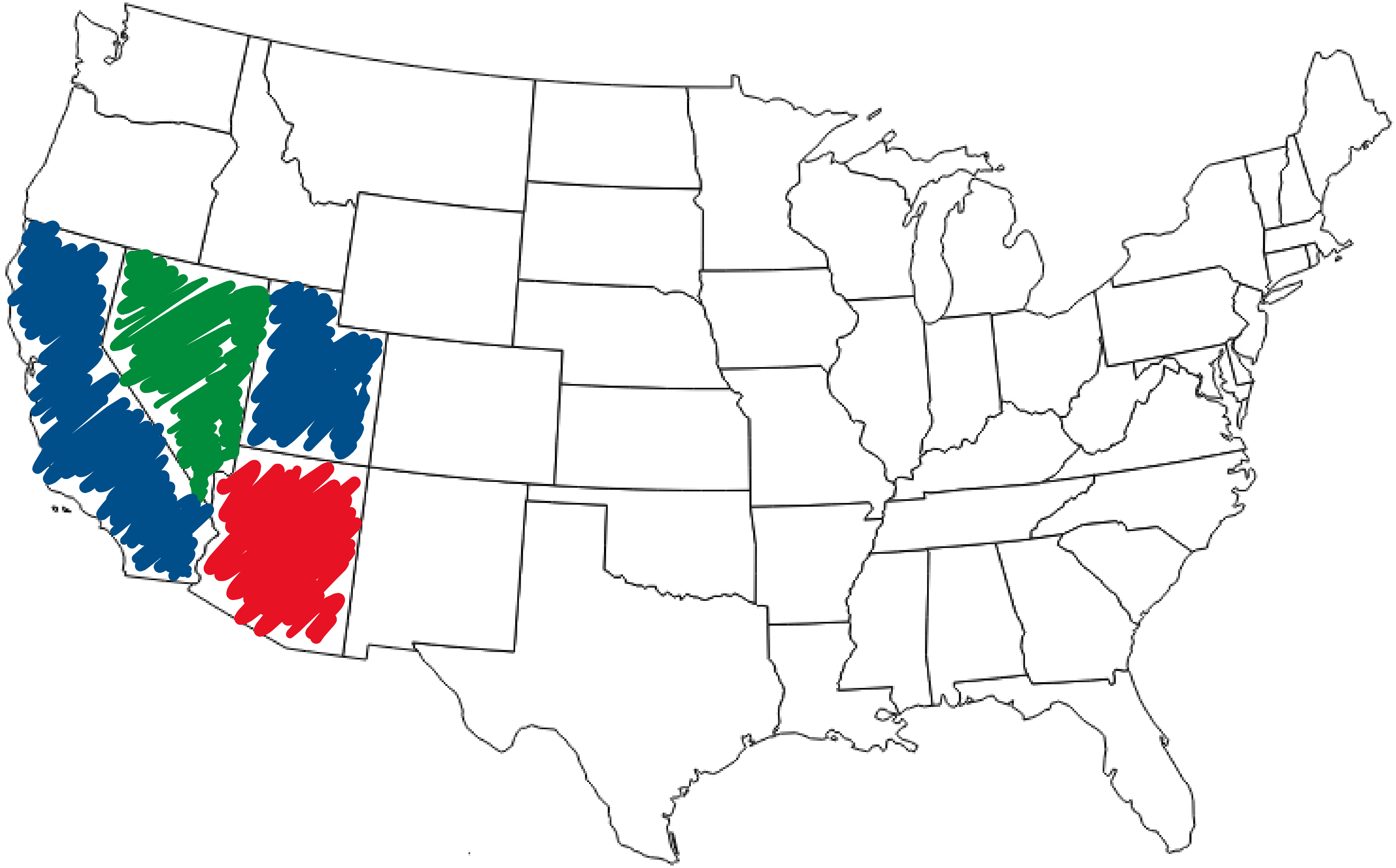
$$K_t \not\preccurlyeq G \to$$
$$\chi(G) = O(t \log\log t)$$

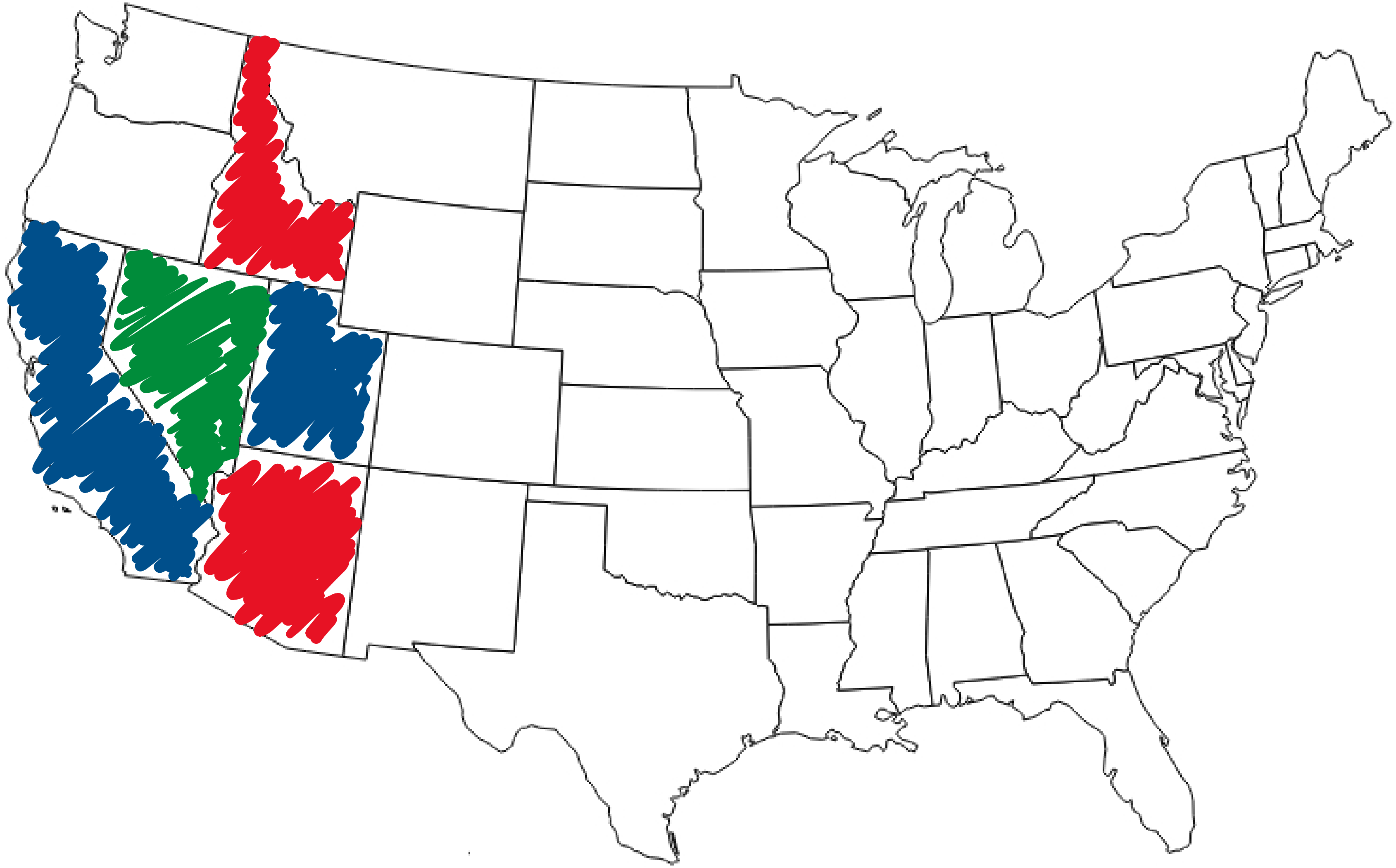
Eden Kuperwasser
TAU, 2026
Student seminar

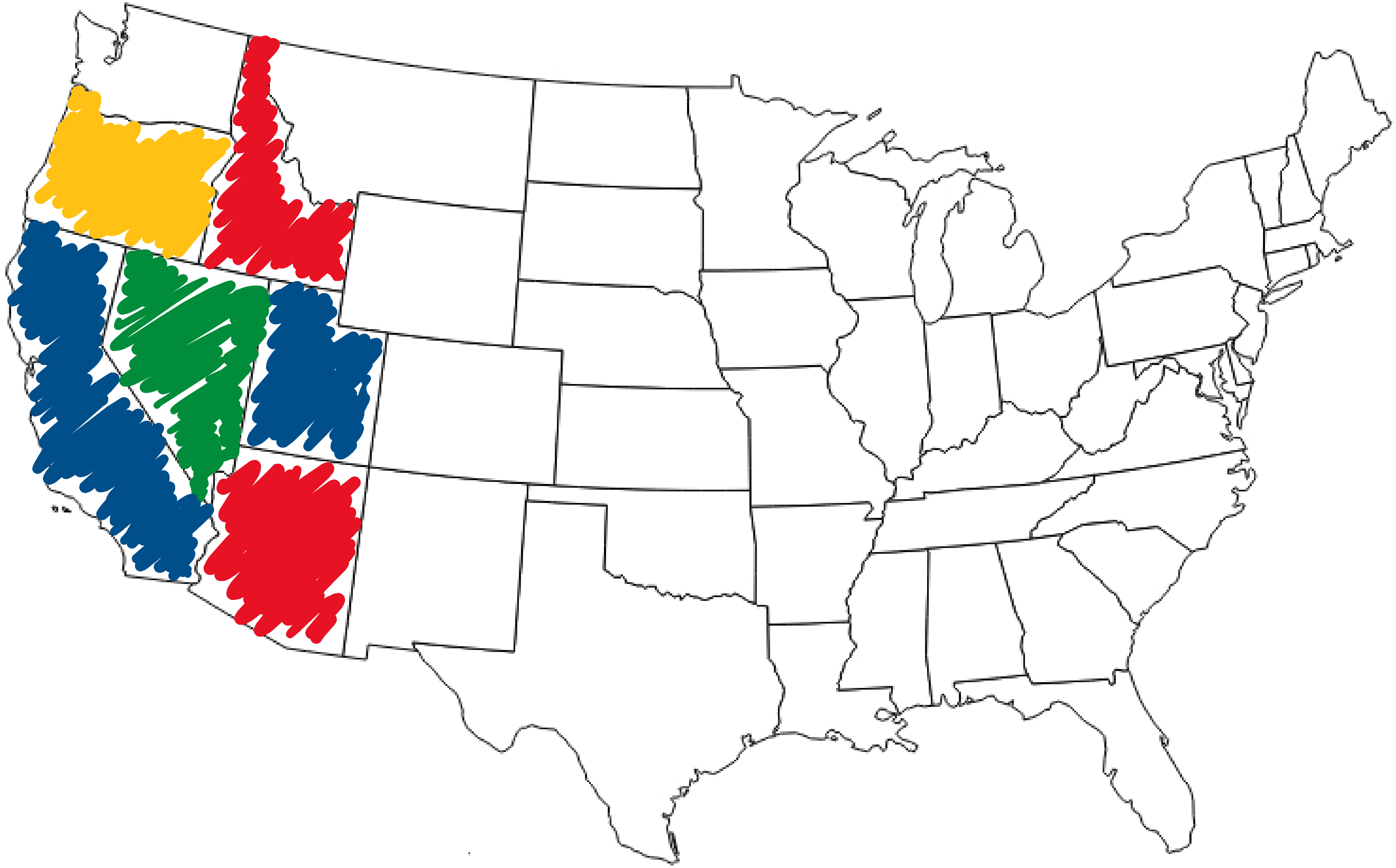


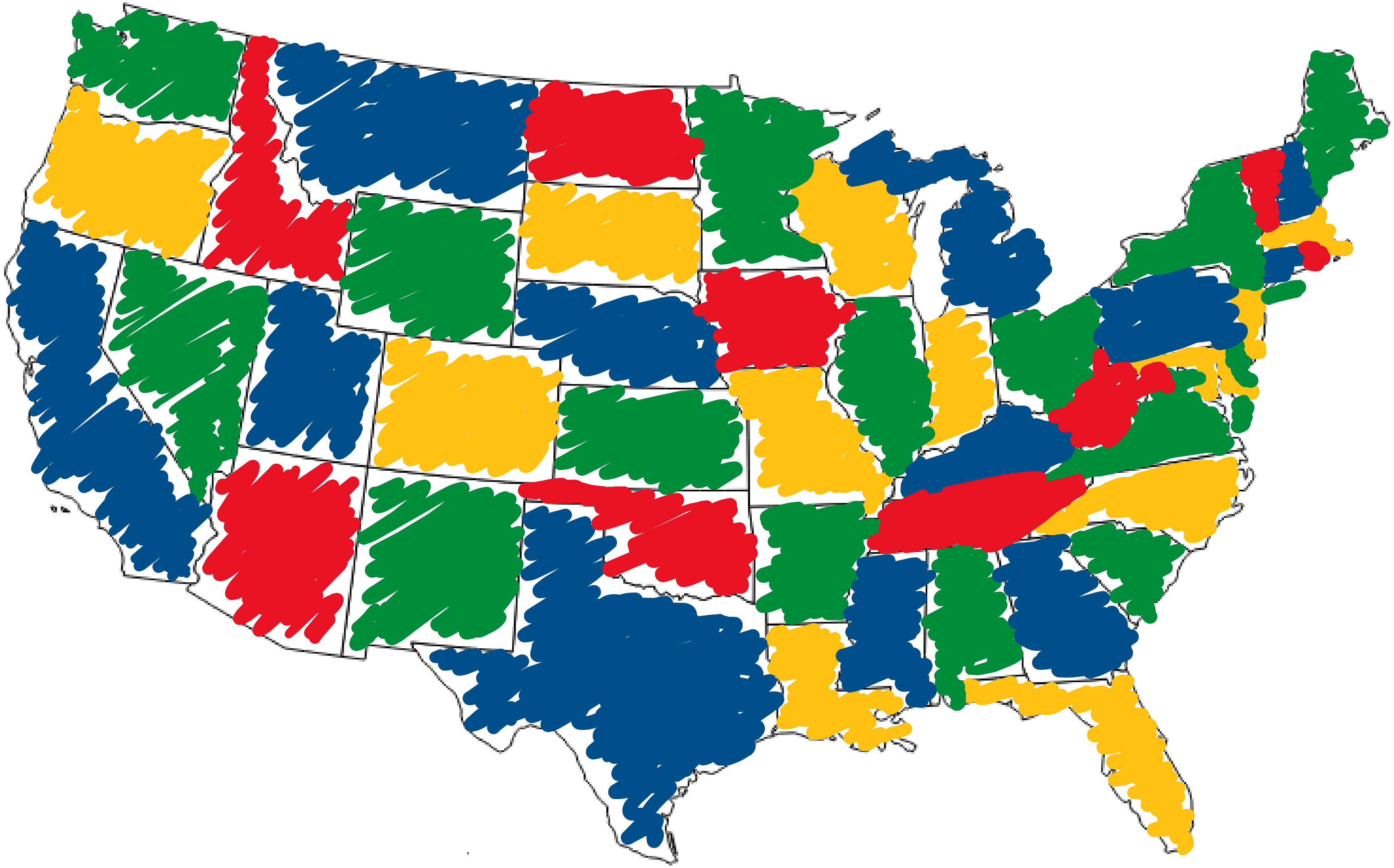












THE 4 COLOR PROBLEM

1852 Francis Guthrie:

Do four colors suffice for every map?

1879 Kempe – Solved!
1880 Tait – Solved!

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THE 4 COLOR PROBLEM

1852 Francis Guthrie:

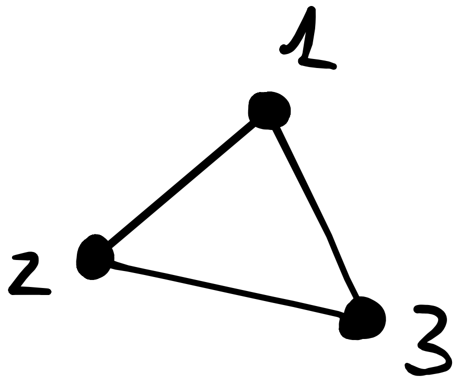
Do four colors suffice for every map?

~~1879 Kempe – Solved!~~ Nope (1890 Heawood).
~~1880 Tait – Solved!~~ Nope (1891 Petersen).

GRAPH THEORY

$K_3 \doteq (\{1,2,3\}, \{\{1,2\},\{1,3\},\{2,3\}\})$

DFN: A **graph** $G = (V, E)$

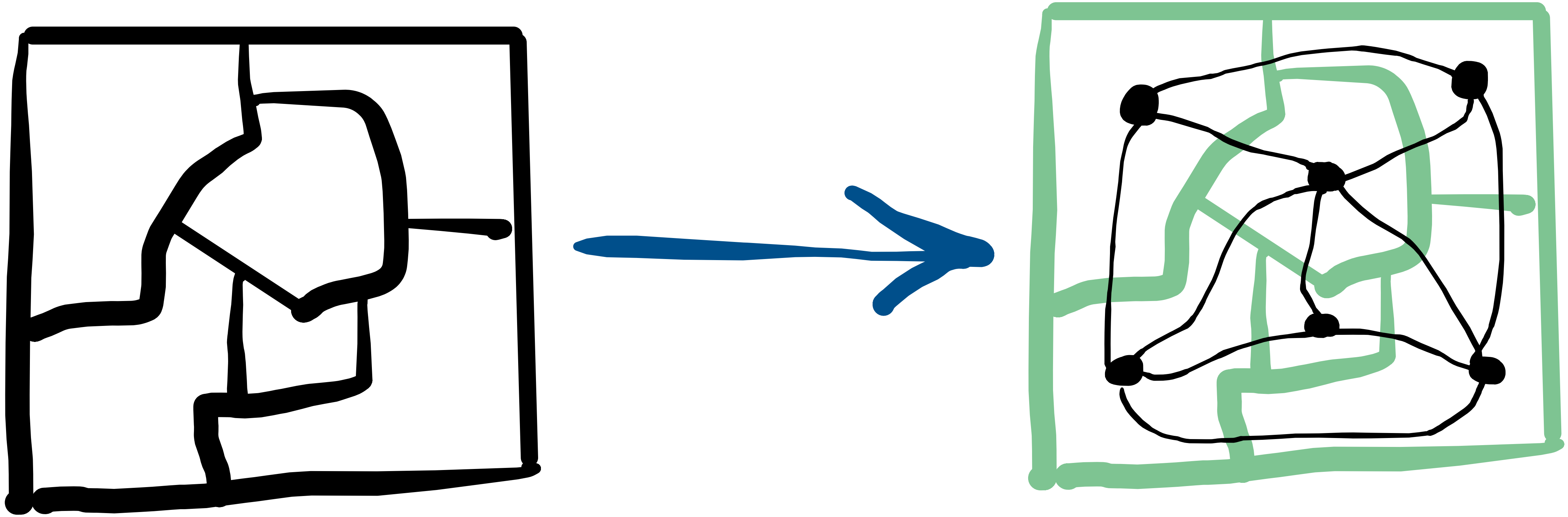


V - **vertices**

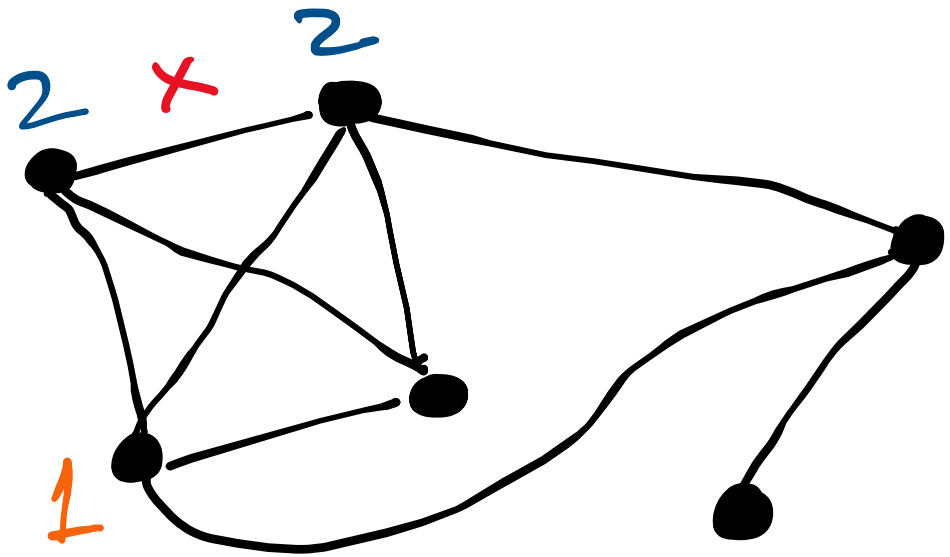
E - **edges**, pairs of vertices.

GRAPH FORMULATION

Vertices = regions
Edges = sharing border



COLORING

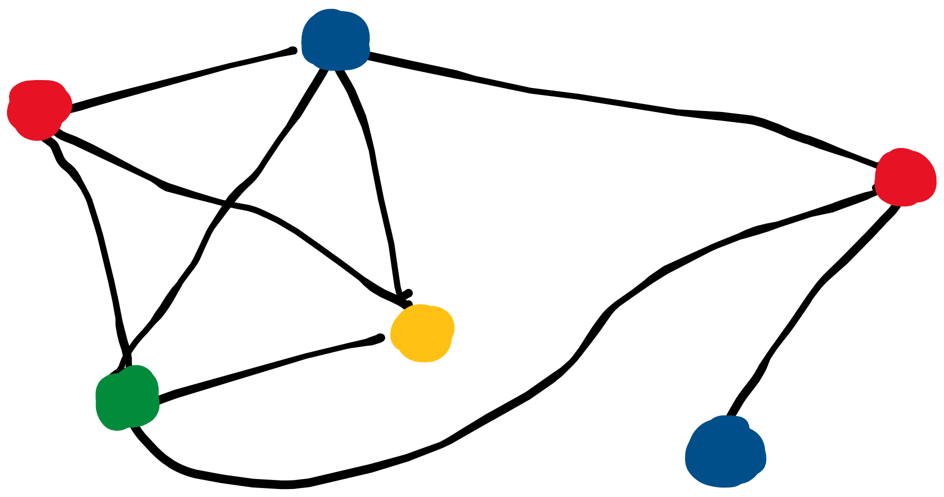


DFN:

Let $G = (V, E)$.

A proper r-**coloring** of G is a function $f\colon V \to \{1, \ldots, r\}$ such that for every edge uv we have $f(u) \neq f(v)$.

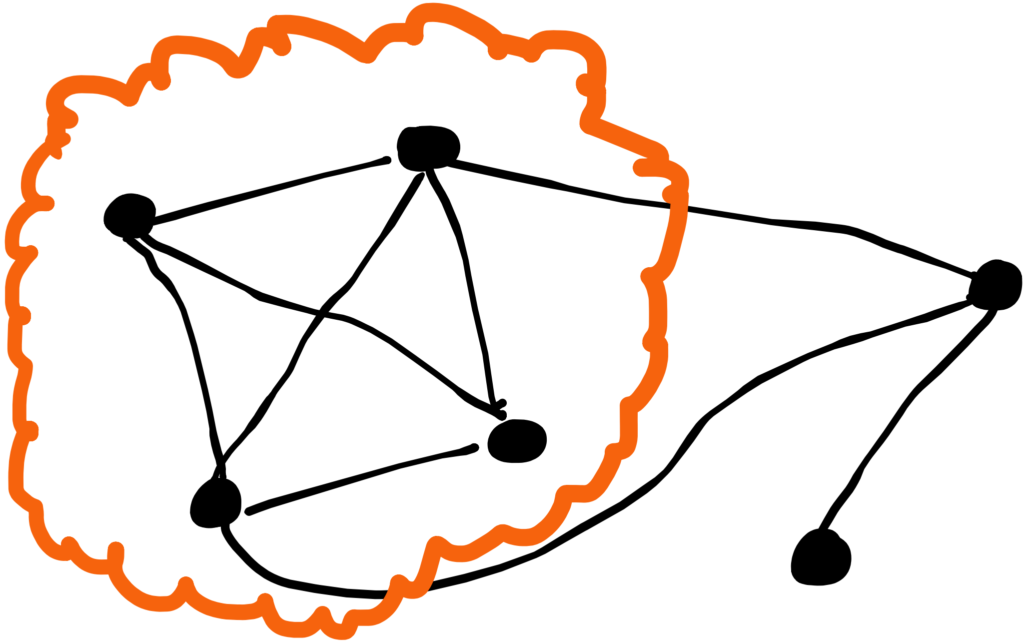
COLORING



Proper coloring with 4 colors

1234

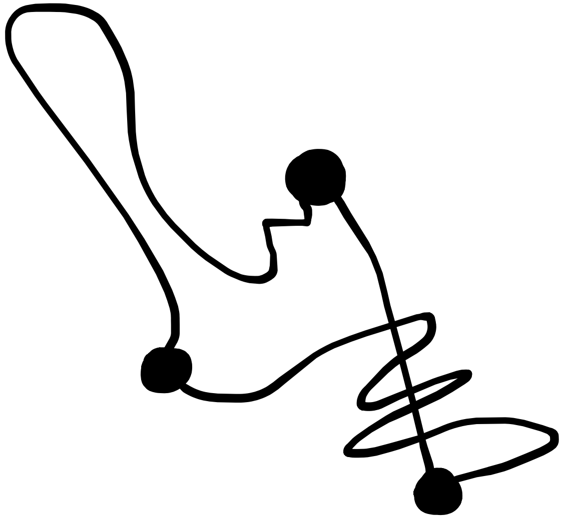
COLORING



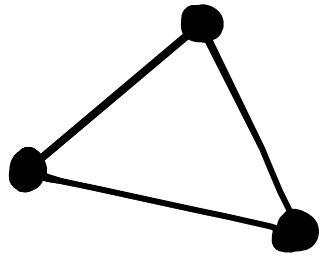
Can we do with three?

PLANAR GRAPHS

DFN: An **embedding** of a graph in a surface.

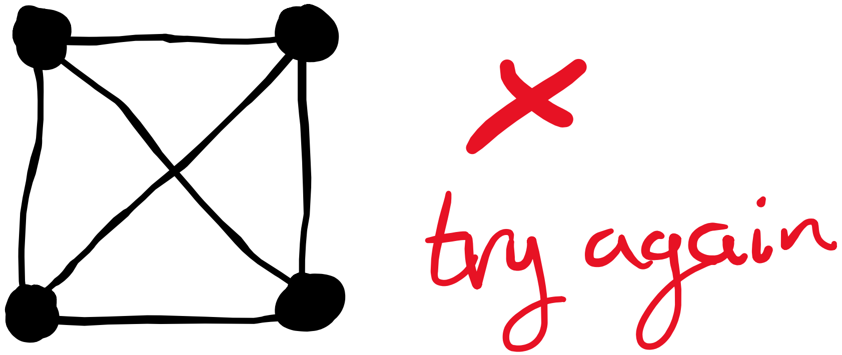


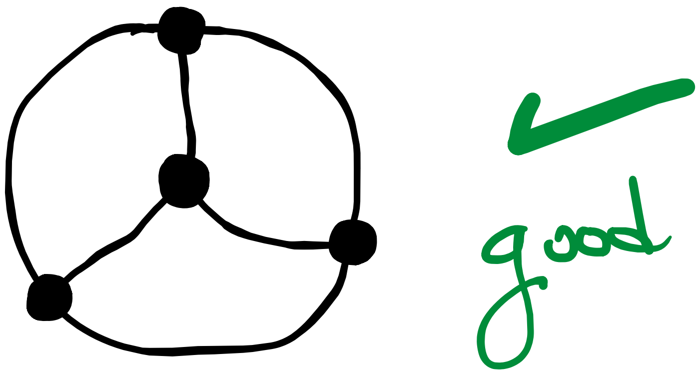
DFN: G is **planar** if it can be embedded in the plane with no crossing edges.



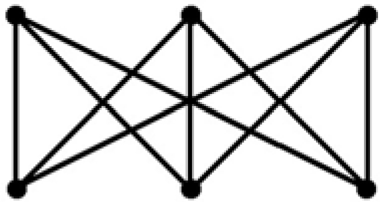
PLANAR GRAPHS – ANOTHER EXAMPLE

Is K_4 planar?



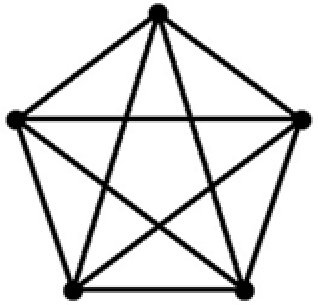


PLANAR GRAPHS – COUNTEREXAMPLES



$K_{3,3}$ is **not** planar.

More importantly, K_5 isn't either ☺ ☺  ☺ ☺



USEFUL LEMMA

LEM: If G is a planar graph then #edges $\leq$ 3 #vertices – 6.

(proof) : Euler characteristic V – E + F = 2 & some fun.

COR: K_5 is not planar. (10 > 3x5 – 6.)

AVERAGE DEGREE

COR: If G planar then its average **degree** < 6.

(proof) sum of degrees $= 2$ #edges.

COR: The 6 Color Theorem!

4 COLOR PROBLEM

1852: birth.

THM (Heawood 1890): 5 colors suffice.

THM (Appel & Haken 1976): 4 colors suffice.

A BIT ABOUT 4CT

One highlight: **Discharging.**

~1,500 configurations, ~40 days computation.

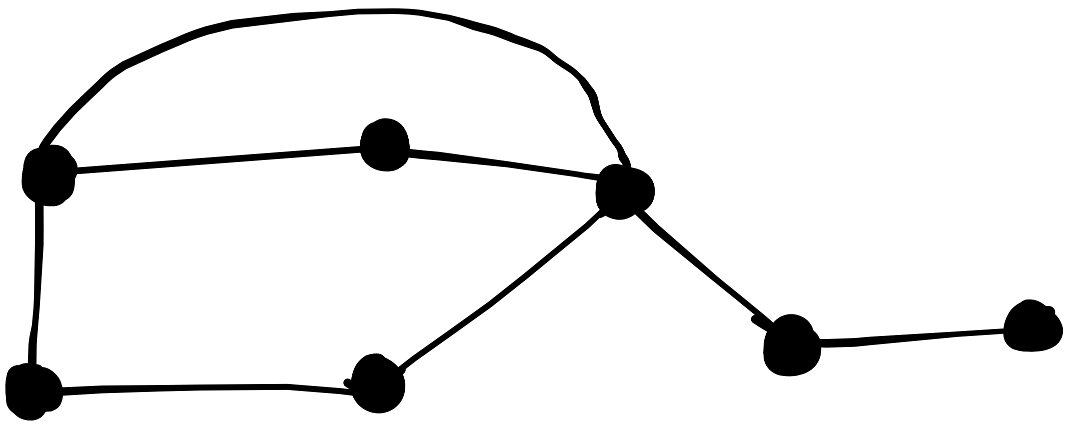
Still no human-verifiable proof ☹

MORE GRAPH THEORY - MINORS

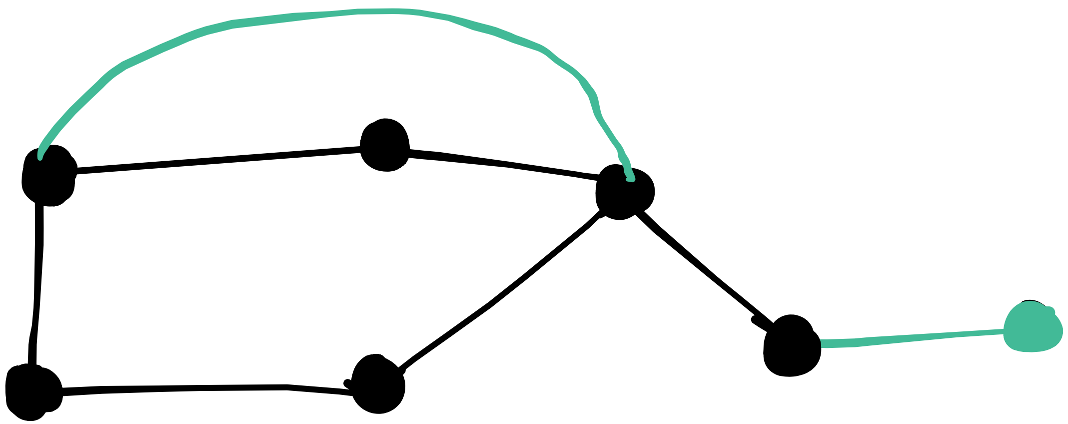
DFN: Let G be a graph. A **minor** of G is a graph achieved by repeated:

- deletion of vertices / edges
- edge contraction

A MINOR EXAMPLE



A MINOR EXAMPLE



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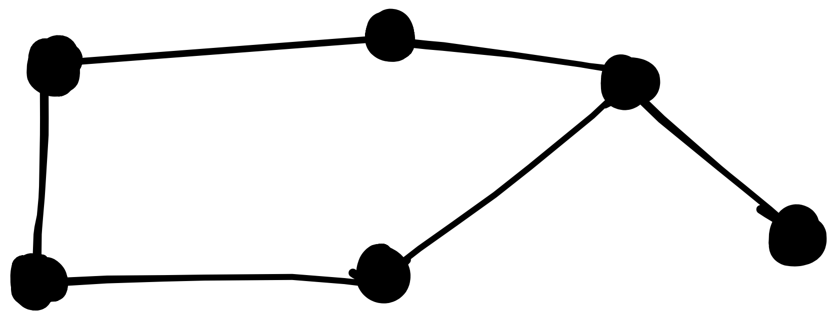

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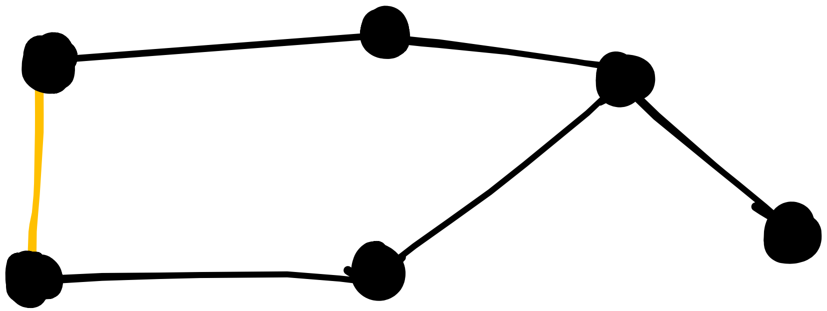

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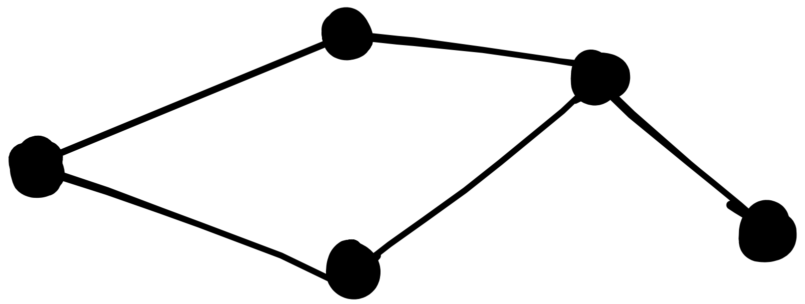

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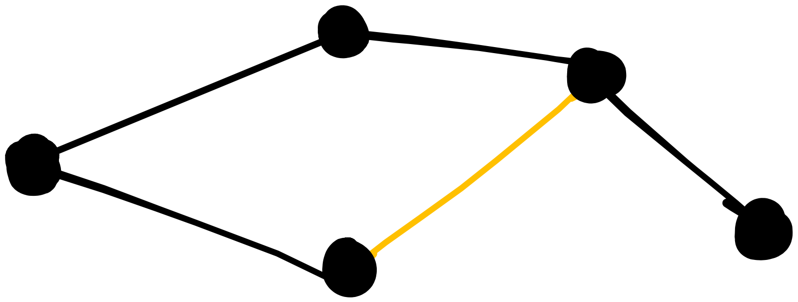

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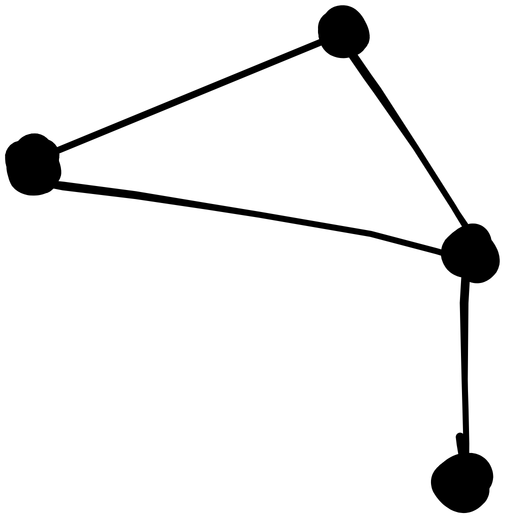

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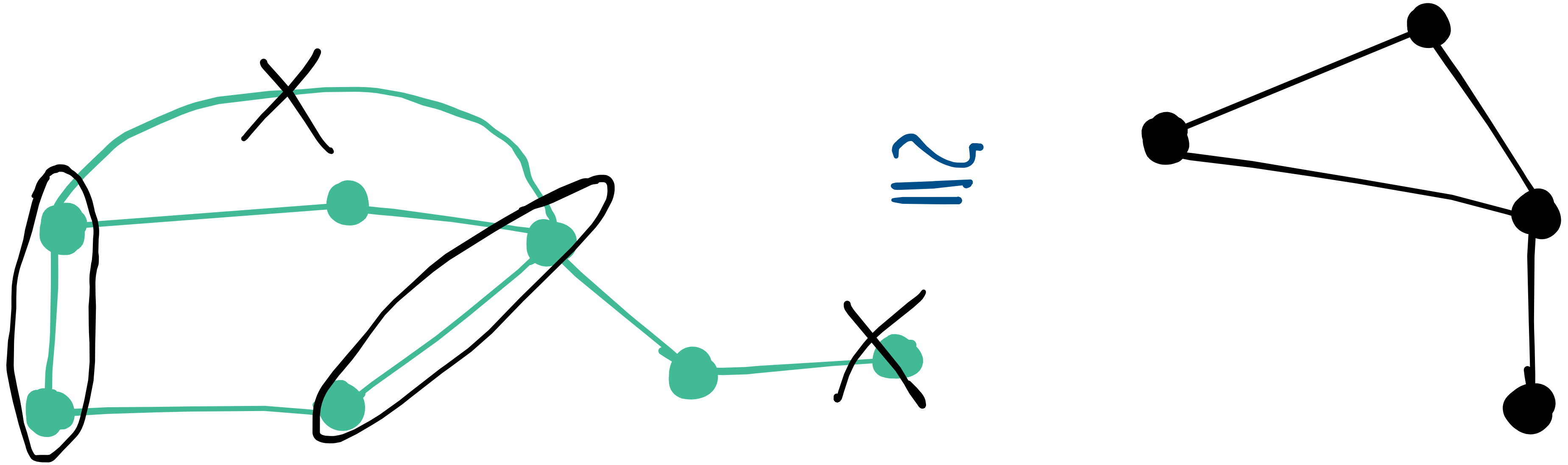


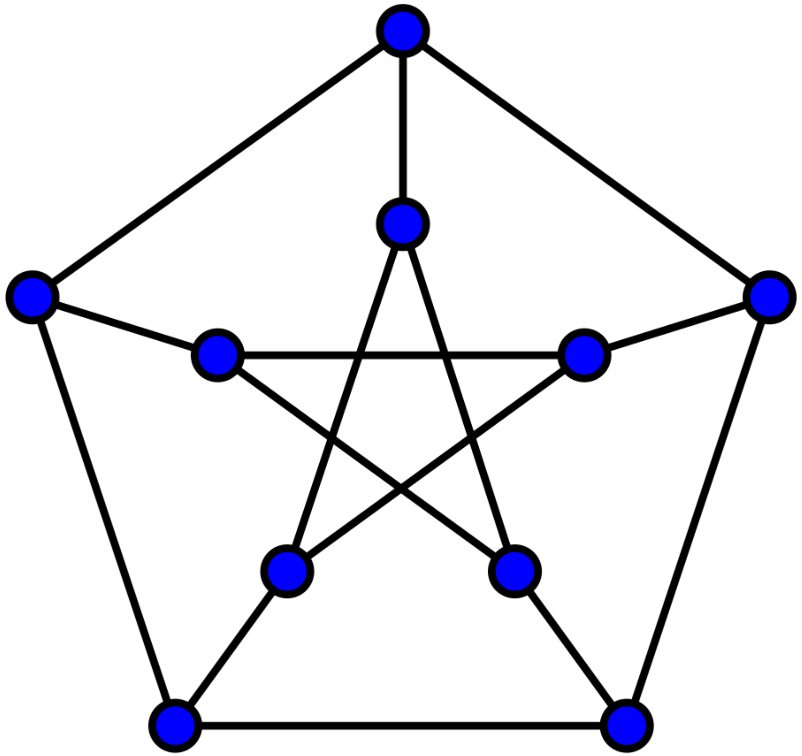
MINORS AND PLANARITY

CLM: Planar graphs are **closed** under minors.

WHICH MEANS:

If G contains a nonplanar minor, it is not planar.

A MAJOR EXAMPLE



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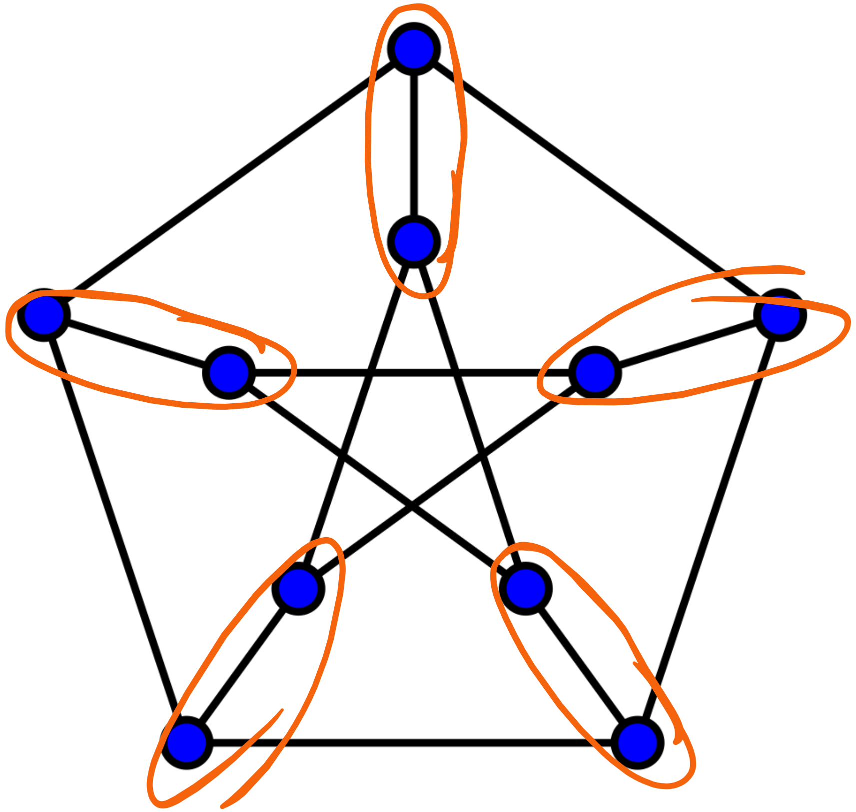


MINORS AND PLANARITY

<u>**THM**</u> (Kuratowski, Wagner 1937):

A graph is planar iff it has no K_5 and no $K_{3,3}$ minors.

A word about the <u>Robertson-Seymour theorem</u>.

HADWIGER'S CONJECTURE

<u>**CONJ**</u> (Hadwiger 1943):

A graph with no K_t minor is $(t-1)$-colorable.

Immediately implies 4CT.

A FEW SMALL CASES

G has no K_t minor.

$t = 1,2,3$: easy.

$t = 4$: not as easy. But true!

$t = 5$: Wagner 1937: Equivalent to 4CT.

$t = 6$: Robertson-Seymour-Thomas 1993: Equiv to 4CT.

CONJECTURE? I HARDLY KNOW HER

Is there any connection between minors and coloring?

For normal subgraphs this **fails miserably**.

ANY CONNECTION

THM (Wagner 1964):

A graph with no K_t minor is 2^t colorable.

(cute proof): To the board!

AVERAGE DEGREES STRIKE BACK

THM (Kostochka, Thomason 1984):

No K_t minor $\Rightarrow$ average degree $\leq C \cdot t\sqrt{\log t}$.

COR: $O\left(t\sqrt{\log t}\right)$ colors suffice.

AVERAGE DEGREE – THE END OF THE ROAD

THM (Bollobás—Catlin—Erdős 1980):

There are graphs that have no K_t minor but have average degree $c \cdot t\sqrt{\log t}$.

(proof): Take a random graph.

BEYOND AVERAGE

THM (Norine—Postle—Song 2023):

$$O\left(t\,(\log t)^{\beta}\right)\text{-colorable where } 1/4 < \beta.$$

Key idea – density increment.

THE STATE OF THE ART

THM (Delcourt—Postle 2024):

$O(t \log \log t)$ colors suffice.

Reduced **Linear Hadwiger** to **coloring small graphs.**

COLORING SMALL GRAPHS

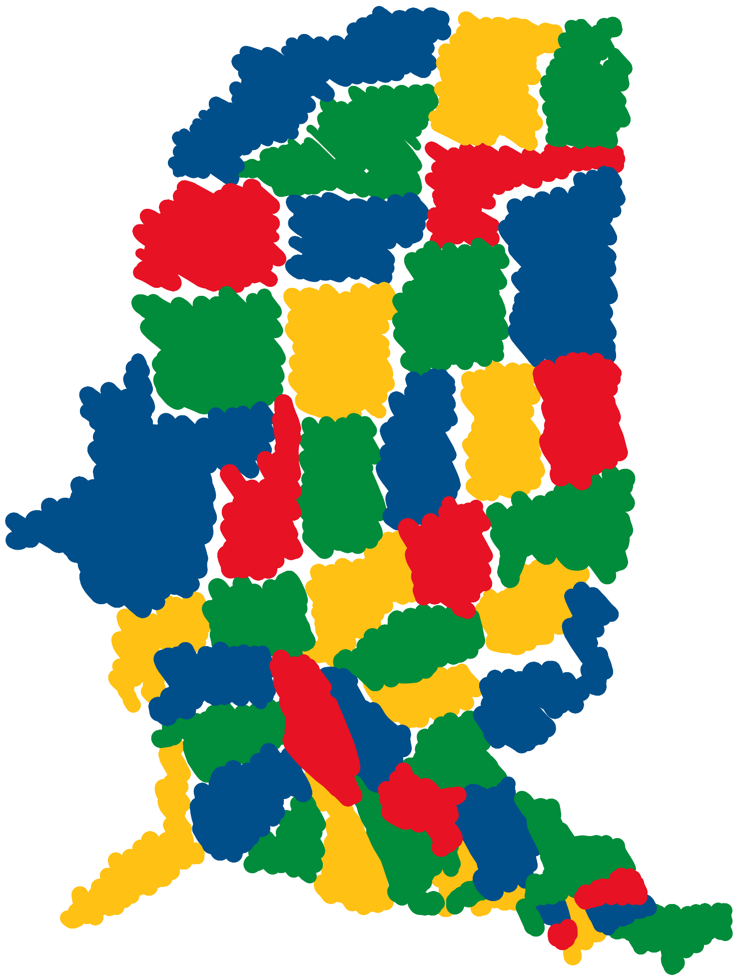
THM (Duchet—Meyniel 1982, Woodall 1987):

Suppose G has no K_t minor. Then we can properly color **half** the vertices of G with t-1 colors.

COR: If G has no K_t minor, we can properly color it with

$$\left(\log_2\left({}^{|V|}/_{t}\right) + 2\right) \cdot t \text{ colors.}$$

THANKS FOR LISTENING!



Questions?